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A Metric-Space Formulation of Newtonian Fields

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Abstract

In the usual curved-space description of gravity, a class of fields is defined which correspond to the fields of Newtonian gravitational theory. Using these fields, the field equations of Newtonian theory are formulated in a four-dimensional metric space. The equations are then modified so that they transform properly under the Lorentz transformation and so that their weak-field approximation is closely analogous to the equations of classical electrodynamics. The resulting equations lead to Newtonian theory in the non-relativistic limit, and they lead rigorously to the Schwarzschild field and to the known relativistic corrections associated with it. Finally, these field equations are compared with Einstein's field equations.

1. *Introduction*

During the last half-century, Einstein's theory has become the most widely accepted description of gravity, but it has not yet led to a generally accepted unification of gravity with the rest of physics. Many attempts have been made to provide such a unification, usually by retaining Einstein's description of gravity in terms of a curved four-space but by generalizing or reinterpreting the rest of the theory. Most of these attempts involve complexities that are at least as great as those of Einstein's theory and are sometimes much greater, even though it could well be argued that Einstein's theory is already far more complex than is really necessary. For example, Newton's theory of gravity gives a very accurate description of most gravitational phenomena in terms of a single potential function, where Einstein's theory requires *ten* metric coefficients to give only a very few detectable corrections to Newton's theory.

This great formal complexity has been accepted partly because of the verifiable corrections that it makes to Newton's theory and partly because it offers a marked philosophical advance over Newtonian theory. For example, Newton's theory predicts the acceleration of bodies relative to a primary inertial system which is fixed relative to the fixed stars, and in this way it relates the motion of the fixed stars directly to the motion of a nearby body even though there is no direct physical connection between them. In Einstein's theory, the motion of a body is described by the geodesic equation, which refers only to the position of the body and to the metric field in its immediate vicinity. The metric field itself is determined from differential equations which do not refer to any remote system such as the fixed stars, and the fixed stars enter into the theory only as boundary conditions. Any change in these boundary conditions affects the motion of nearby bodies only because it is propagated through the intervening field, which then provides a physical connection between the body under observation and the fixed stars.

Of course, it is always possible to formulate Newton's theory in generalized space-time coordinates so that the motion of the primary inertial system is described by field quantities which can be determined from differential equations somewhat analogous to Einstein's (Havas, 1964). However, the introduction of additional field quantities which can *always* be removed by an appropriate coordinatization seems somewhat artificial. Also, theories of this type involve at least as many unknown functions as Einstein's theory, and hence do nothing to reduce its complexity, while at the same time they predict none of Einstein's verifiable corrections to Newtonian theory. Thus, for both experimental and philosophical reasons, it seems desirable to describe gravity in terms of Einstein's curved Riemannian four-space, in which particles and light rays are assumed to move along geodesics. However, it will be shown in this paper that the complexity of Einstein's theory can be greatly reduced by considering a much smaller class of metric fields than was originally considered by Einstein. This procedure keeps the philosophical advantage of Einstein's theory and will also keep most of its verifiable corrections to Newtonian theory if only the Schwarzschild field is included in the allowed class of metrics.

The first step will be to consider only the metrics which correspond to the fields of Newtonian gravitational theory and to use these metrics to formulate the Newtonian field equations within the framework of a curved metric four-space. It will then be shown that the resulting field equations can be made to transform properly under the Lorentz group if the allowed class of fields is permitted to be slightly larger than just the Newtonian fields but still much smaller than the class of fields usually considered in Einstein's theory. The metrics of interest will be those in which the three-dimensional geometry is Euclidean, with the result that much of the formal complexity usually associated with Einstein's theory is removed. However, its philosophical advantages over Newtonian theory are kept, as are also all of its corrections to Newtonian theory in the Schwarzschild field. Furthermore, the Lorentz-invariant extension of the Newtonian field equations can be made more closely analogous to Maxwellian electrodynamics than are Einstein's field equations, a result which is highly desirable in any attempt to unify physics.

Finally, the field equations derived here will be compared directly with Einstein's equations.

2. *Newtonian Fields*

It will be assumed here that any gravitational field can be described by a curved Riemannian four-space whose metric coefficients $g_{\alpha\beta}$ determine the local time $d\tau$ measured by a moving clock by means of the relation

$$-c^2\,d\tau^2 = g_{\alpha\beta}\,dx_\alpha\,dx_\beta \tag{2.1}$$

where x_α are any four coordinates which span the space and c is the velocity of light. In this equation, and throughout this paper, the notation is adopted that Greek indices run from one to four and repeated Greek indices are summed from 1 to 4, while Roman indices will run from 1 to 3 and repeated Roman indices will be summed from 1 to 3. It will be further assumed throughout this paper that the paths of test particles and light rays are geodesics of the metric $g_{\alpha\beta}$. These assumptions are common and need no further discussion.

Newtonian gravitational theory can be described in this framework by considering a particular class of metrics which have been shown previously to correspond to the fields of Newton's theory (Kirkwood, 1970). The first requirement placed on these metrics is that there must exist a Newtonian time function t such that the three-dimensional geometry on the surface defined by a constant value of t is Euclidean. For such fields, the time-like coordinate x_4 can be chosen to equal t and the remaining three coordinates x_i can be chosen to be Cartesian coordinates in the Euclidean three-space. In these coordinates the metric coefficients $g_{\alpha\beta}$ satisfy the three-dimensional relation

$$g_{ij} = \delta_{ij} \tag{2.2}$$

where δ_{ij} is the identity.

The motion of a particle in such a field is governed by the geodesic equation, which can be written in the form

$$g_{\alpha\beta}\frac{d^2x_\beta}{d\tau^2} = -(\beta\gamma,\alpha)\frac{dx_\beta}{d\tau}\frac{dx_\gamma}{d\tau} \tag{2.3}$$

If this equation is multiplied by $dx_\alpha/d\tau$ it reduces to

$$\frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx_\alpha}{d\tau}\frac{dx_\beta}{d\tau}\right) = 0$$

which is satisfied identically because of equation (2.1). Thus it is clear that if equation (2.3) is satisfied for $\alpha = 1, 2, 3$ and if $dx_4/d\tau \neq 0$, then equation (2.3) will automatically be satisfied for $\alpha = 4$ by virtue of equation (2.1), so that it is sufficient to consider equation (2.3) only for the three values of α of 1, 2 and 3. Noting that the metric coefficients are assumed to satisfy equation (2.2), it is then readily shown that equation (2.3) can be written in the completely three-dimensional form

$$\frac{d^2x_i}{d\tau^2} = \left[\left(\frac{\partial g_{j4}}{\partial x_i} - \frac{\partial g_{i4}}{\partial x_j}\right)\frac{dx_j}{dx_4} - \frac{\partial g_{i4}}{\partial x_4} + \frac{1}{2}\frac{\partial g_{44}}{\partial x_i}\right]\left(\frac{dx_4}{d\tau}\right)^2 - g_{i4}\frac{d^2x_4}{d\tau^2} \tag{2.4}$$

This is the exact equation of motion of a particle in a field in which equation (2.2) is satisfied. The usual equation of motion of Newton's theory is the non-relativistic approximation of equation (2.4), in which the time measured by a moving clock is approximately equal to the time variable x_4, so that $dx_4/d\tau = 1$, $d^2x_4/d\tau^2 = 0$, and $d^2x_i/d\tau^2$ is the acceleration of the moving particle, which is then given by the quantity in brackets in equation (2.4). This quantity is the gravitational force per unit mass, which in Newton's theory is independent of the particle velocity and equal to the negative gradient of the Newtonian potential V. If the bracketed quantity in equation (2.4) is to be independent of the particle velocity, it must be that the quantity in parentheses vanishes, which is equivalent to the requirement that a function β exists such that

$$g_{i4} = -\frac{\partial\beta}{\partial x_i} \tag{2.5}$$

Then the bracketed quantity takes the form $-\partial V/\partial x_i$ if

$$V = -\frac{\partial\beta}{\partial x_4} - \frac{g_{44}}{2} - \frac{c^2}{2} \tag{2.6}$$

where the constant $-c^2/2$ is added to make $V \to 0$ in the region far from any masses, where $\partial\beta/\partial x_4 \to 0$ and $g_{44} \to -c^2$. From equations (2.2), (2.5) and (2.6), it is seen that the metric four-space is of the type described by Newton's theory if there exist two functions β and V and a set of coordinates x_α such that the metric tensor in the coordinates x_α takes the form

$$g_{ij} = \delta_{ij} \tag{2.7a}$$

$$g_{i4} = -\frac{\partial\beta}{\partial x_i} \tag{2.7b}$$

$$g_{44} = -2V - 2\frac{\partial\beta}{\partial x_4} - c^2 \tag{2.7c}$$

Fields of this type will be called Newtonian fields. It has been shown previously (Kirkwood, 1970) that if $V = -KM/r$ and $\beta = -(8KMr)^{1/2}$, where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, the metric coefficients of equations (2.7a)–(2.7c) can be reduced to the Schwarzschild field. *Thus, in the non-relativistic approximation, the Newtonian fields give rise to all of Newton's theory of gravity, and when the exact geodesic equations are used to describe the motion of particles or light rays, the Newtonian fields yield all of the relativistic corrections that have been verified experimentally in the Schwarzschild field.* From this it is clear that the Newtonian fields are a very important class of fields, and might conceivably be the only fields that occur in nature.

The possibility that Newtonian fields may admit of an invariance similar to the usual Lorentz invariance of the electromagnetic field has been investigated previously (Kirkwood, 1970). A generalized Lorentz transformation has been defined in such a way that it makes the intrinsic properties of space and time as nearly invariant as possible, while the gravitational field quantities are allowed to transform in any way that is convenient, following the pattern of the usual Lorentz invariance of electromagnetism. The resulting transformation group leaves the time variable t invariant, so that it cannot be the time-like coordinate in more than one Lorentz frame, and it is at least possible that it may not be the time coordinate in *any* Lorentz frame. This invariant time-like function, similar to Newton's universal time, will be seen to play an important role in the formulation of gravity that will be given here. The existence of such an invariant time function does not conflict with the Lorentz invariance of the theory; in fact, it arises quite naturally from it.

3. *The Field Equations*

It has been shown that the gravitational fields of Newton's theory are fields for which there exist coordinates in which the metric coefficients can be expressed in the form of equations (2.7a)–(2.7c). To complete Newtonian theory, it is only necessary to add the requirement that in these coordinates the Newtonian potential function V satisfies Poisson's equation,

$$\nabla^2 V = 4\pi K\mu \tag{3.1}$$

where ∇^2 is the three-dimensional Laplacian operator, K is the gravitational constant and μ is the density of the mass that is producing the field. All of these conditions will now be formulated into a set of field equations that a curved four-space must satisfy if it is to describe one of the gravitational fields of Newtonian theory.

If the field is described by the metric coefficients $g_{\alpha\beta}$, then the first requirement that the $g_{\alpha\beta}$ must satisfy if the field is to be Newtonian is that there must exist a time-like function t such that the three-dimensional geometry determined by $g_{\alpha\beta}$ on a surface on which t is constant is Euclidean. The condition that t is time-like is that

$$g^{\alpha\beta}\frac{\partial t}{\partial x_\alpha}\frac{\partial t}{\partial x_\beta} < 0$$

from which it follows that there is a positive real function α such that

$$g^{\alpha\beta}\frac{\partial t}{\partial x_\alpha}\frac{\partial t}{\partial x_\beta} = -\frac{1}{\alpha^2} \tag{3.2}$$

Letting

$$t^\alpha \equiv g^{\alpha\beta}\frac{\partial t}{\partial x_\beta} \tag{3.3}$$

and defining the tensor

$$h^{\alpha\beta} \equiv g^{\alpha\beta} + \alpha^2 t^\alpha t^\beta \tag{3.4}$$

it is seen from equations (3.2) and (3.3) that

$$h^{\alpha\beta}\frac{\partial t}{\partial x_\beta} = 0 \tag{3.5}$$

The significance of the tensor $h^{\alpha\beta}$ is easily seen in a system of coordinates in which $x_4 = t$, where equation (3.5) shows that $h^{\alpha 4}$ vanishes. The remaining components h^{ij} in these coordinates are the reciprocal of the three-dimensional metric g_{ij}, as can be shown by multiplying equation (3.4) by $g_{\beta\gamma}$ and observing that for $\alpha, \gamma = 1, 2, 3$ the resulting relation reduces to $h^{ij} g_{jk} = \delta_k{}^i$. The condition that the geometry must be Euclidean in the three-space defined by a given value of t is that the curvature tensor in this space must vanish, which, in a three-space, is equivalent to requiring that the contracted curvature tensor must vanish. Let the coordinates be chosen so that $x_4 = t$ and let the curvature tensor in the three-space of x_1, x_2 and x_3 be denoted by ρ_{ijkl} and its contraction by ρ_{il}, so that

$$\rho_{il} = h^{jk} \rho_{ijkl} \tag{3.6}$$

where h^{jk} is the three-dimensional reciprocal of g_{ij} defined above. Then the desired condition that a constant value of t (or x_4) will define a Euclidean three-space is that

$$\rho_{il} = 0 \tag{3.7}$$

This can be expressed in four-dimensional notation by extending ρ_{ijkl} and ρ_{il} into four dimensions in the following way. First, it is noted that in coordinates in which $x_4 = t$, ρ_{ijkl} can be written

$$\rho_{ijkl} = \frac{\partial}{\partial x_k}(jl, i) - \frac{\partial}{\partial x_l}(jk, i) + h^{mn}[(jk, m)(il, n) - (jl, m)(ik, n)] \tag{3.8}$$

where the quantities (jl, i) are the usual three-dimensional Christoffel symbols. However, since $x_4 = t$, the quantities $h^{\alpha 4}$ defined above vanish, and the sums over m and n in equation (3.8) can be extended from 1 to 4 without altering the equation. Then, defining the four-dimensional quantity

$$\rho_{\alpha\beta\gamma\delta} \equiv \frac{\partial}{\partial x_\gamma}(\beta\delta, \alpha) - \frac{\partial}{\partial x_\delta}(\beta\gamma, \alpha) + h^{\lambda\mu}[(\beta\gamma, \lambda)(\alpha\delta, \mu) - (\beta\delta, \lambda)(\alpha\gamma, \mu)] \tag{3.9}$$

where $(\beta\delta, \alpha)$ is the four-dimensional Christoffel symbol and where $h^{\lambda\mu}$ is given by equation (3.4), it is seen that $\rho_{\alpha\beta\gamma\delta}$ equals ρ_{ijkl} when $x_4 = t$ and the indices run from 1 to 3. Writing $h^{\lambda\mu}$ in the form given by equation (3.4) and noting that in coordinates in which $x_4 = t$ the second covariant derivative of t can be written

$$t_{;\alpha\beta} \equiv \frac{\partial^2 t}{\partial x_\alpha \, \partial x_\beta} - (\alpha\beta, \gamma)\, t^\gamma = -(\alpha\beta, \gamma)\, t^\gamma$$

it is found that equation (3.9) can be written

$$\rho_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \alpha^2(t_{;\beta\gamma}\, t_{;\alpha\delta} - t_{;\beta\delta}\, t_{;\alpha\gamma}) \tag{3.10}$$

where $R_{\alpha\beta\gamma\delta}$ is the four-dimensional curvature tensor constructed from $g_{\alpha\beta}$. If $\rho_{\alpha\beta\gamma\delta}$ is defined in arbitrary coordinates by the usual rule for the transformation of a covariant tensor, then equation (3.10) is a tensor equation and holds in any coordinates, because α^2 is an invariant by equation (3.2).

Similarly, ρ_{ii} can be extended into four dimensions by noting that the sums over j and k in equation (3.6) can be extended from 1 to 4 without affecting the equation, because $h^{\alpha 4} = 0$ when $x_4 = t$. Then a four-dimensional tensor $\rho_{\alpha\beta}$ can be defined in these coordinates by the relation

$$\rho_{\alpha\delta} = h^{\beta\gamma} \rho_{\alpha\beta\gamma\delta} \tag{3.11}$$

where $\rho_{\alpha\beta\gamma\delta}$ is given by equation (3.9). The spatial components of $\rho_{\alpha\delta}$ will be the ρ_{ii} of equation (3.6), and the condition that the three-space associated with a given value of t will be Euclidean will be that $\rho_{ii} = 0$.

If $\rho_{ii} = 0$ it will be possible to choose the three spatial coordinates x_i so that $g_{ij} = \delta_{ij}$. In these coordinates h^{ij} will equal δ_{ij} and $h^{\alpha 4}$ will vanish, so that the remaining components of $\rho_{\alpha\delta}$, namely the components $\rho_{\alpha 4}$, can be expressed from equations (3.9) and (3.11) in the form

$$\rho_{\alpha 4} = \frac{\partial}{\partial x_i}(i4,\alpha) - \frac{\partial}{\partial x_4}(ii,\alpha) + (ii,j)(\alpha 4,j) - (i4,j)(\alpha i,j)$$

If the Christoffel symbols are evaluated for a metric for which $g_{ij} = \delta_{ij}$ then the three-dimensional (ij,k) vanishes, and the above equation can be written

$$\rho_{\alpha 4} = \begin{cases} \dfrac{\partial}{\partial x_i}(i4,\alpha) & (\alpha = 1,2,3) \\[2ex] \dfrac{\partial}{\partial x_i}(i4,4) - \dfrac{\partial}{\partial x_4}(ii,4) - (i4,j)(4i,j) & (\alpha = 4) \end{cases}$$

which finally becomes

$$\rho_{i4} = -\frac{1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial g_{j4}}{\partial x_i} - \frac{\partial g_{i4}}{\partial x_j}\right) \tag{3.12a}$$

$$\rho_{44} = \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_i \partial x_i} - \frac{\partial^2 g_{i4}}{\partial x_4 \partial x_i} - \frac{1}{4}\left(\frac{\partial g_{i4}}{\partial x_j} - \frac{\partial g_{j4}}{\partial x_i}\right)\left(\frac{\partial g_{i4}}{\partial x_j} - \frac{\partial g_{j4}}{\partial x_i}\right) \tag{3.12b}$$

In addition to the requirement that the three-dimensional geometry be Euclidean in coordinates in which $x_4 = t$, a Newtonian field must satisfy equation (2.7b) in these coordinates, which is equivalent to requiring that

$$\frac{\partial g_{i4}}{\partial x_j} - \frac{\partial g_{j4}}{\partial x_i} = 0 \tag{3.13}$$

From equation (3.12a) this obviously implies that

$$\rho_{i4} = 0 \tag{3.14}$$

Not only is equation (3.14) a necessary result of equation (3.13), but, when used with appropriate boundary conditions, it is sufficient to insure that equation (3.13) *must* be satisfied. To show this, it is convenient to define a three-dimensional vector **a** whose components are cg_{i4}. Then the condition that $\rho_{i4} = 0$ can be written from equation (3.12a) in the form $\nabla x \nabla x\mathbf{a} = 0$.

It is known that if the divergence and curl of a vector vanish in any volume and if the normal component of the vector vanishes on the surface bounding that volume, then the vector must vanish everywhere in the volume. Since $\nabla . (\nabla x \mathbf{a})$ vanishes identically, it is only necessary to require that $\nabla x \nabla x \mathbf{a}$ vanish and that $\nabla x \mathbf{a}$ satisfy appropriate boundary conditions to insure that $\nabla x \mathbf{a}$ vanishes everywhere, and hence that equation (3.13) is satisfied. Thus equations (3.7) and (3.14), when taken with appropriate boundary conditions, are both necessary and sufficient to insure that the field is Newtonian.

Finally, if the field is Newtonian, so that g_{i4} and g_{44} are given by equations (2.7b) and (2.7c), equation (3.12b) becomes

$$\rho_{44} = -\nabla^2 V$$

If it is further required that V must satisfy Poisson's equation, then from equation (3.1) it is only necessary to let $\rho_{44} = -4\pi K\mu$. *A complete and exact description of the fields of Newtonian theory is given by this relation, equations* (3.7) *and* (3.14), *and appropriate boundary conditions.* The field equations then are

$$\rho_{ij} = 0 \tag{3.15a}$$

$$\rho_{i4} = 0 \tag{3.15b}$$

$$\rho_{44} = -4\pi K\mu \tag{3.15c}$$

These equations can be easily expressed in arbitrary space-time coordinates by writing $\rho_{\alpha\delta}$ in the form of equation (3.11), where $\rho_{\alpha\beta\gamma\delta}$ is given by equation (3.10) and $h^{\beta\gamma}$ is given by equation (3.4). This gives

$$\rho_{\alpha\delta} = (g^{\beta\gamma} + \alpha^2 t^\beta t^\gamma)[R_{\alpha\beta\gamma\delta} + \alpha^2(t_{;\beta\gamma} t_{;\alpha\delta} - t_{;\beta\delta} t_{;\alpha\gamma})] \tag{3.16}$$

where α^2 is given by equation (3.2). Since t is an invariant function, the right-hand side of this equation is obviously a tensor, so that the quantities $\rho_{\alpha\delta}$ defined in an arbitrary system of coordinates by equation (3.16) will transform as a tensor and will reduce to the quantities on the left-hand side of equations (3.15a)–(3.15c) in the particular coordinates in which $x_4 = t$. The right-hand side of equations (3.15a)–(3.15c) can be written in tensor form by noting that in these coordinates $\partial t/\partial x_\alpha = (0001)$, and equations (3.15a)–(3.15c) can be written in the four-dimensional form

$$\rho_{\alpha\beta} = -4\pi K\mu \frac{\partial t}{\partial x_\alpha} \frac{\partial t}{\partial x_\beta} \tag{3.17}$$

where $\rho_{\alpha\beta}$ is given by equation (3.16). This form of the equations holds in any coordinate system.

4. *A Lorentz Invariant Extension of Newtonian Theory*

The investigation of the Lorentz invariance of Newtonian fields that has been reported previously (Kirkwood, 1970) shows that the generalized form of the Lorentz transformation treats the three functions β, V and t as invariants and leaves the functional form of the metric coefficients $g_{\alpha\beta}$ the same in all Lorentz frames when $g_{\alpha\beta}$ is expressed in terms of β, V and t and their partial derivatives. It is seen from equation (3.16) that $\rho_{\alpha\beta}$ can be expressed in arbitrary space-time coordinates in terms of $g_{\alpha\beta}$ and t and their partial derivatives, so that $\rho_{\alpha\beta}$ will have the same functional form in all Lorentz frames when it is expressed in terms of β, V and t and their partial derivatives. Since t is invariant, the right-hand side of equation (3.17) will have the same form in any Lorentz frame if $K\mu$ is transformed as an invariant function. As a result, the functional form of equation (3.17) will be the same in all Lorentz frames, and the theory will be Lorentz invariant if $K\mu$ is invariant under the generalized Lorentz transformation.

However, it is usually assumed that K is a constant which has the same value in all Lorentz frames and that μ is the same mass density that gives rise to inertia, in which case it is well known that μ is not invariant under the Lorentz transformation and $K\mu$ is not an invariant function. In the description of fluid mechanics in the special theory, the inertial mass density appears as one component of the stress-energy-momentum tensor $P^{\alpha\beta}$ associated with the ponderable matter. This tensor vanishes where there is no matter, and is defined in coordinates in which $x_4 = t$ by the relations

$$P^{ij} = S^{ij} + \mu u^i u^j \tag{4.1a}$$

$$P^{i4} = \mu u^i \tag{4.1b}$$

$$P^{44} = \mu \tag{4.1c}$$

where μ is the mass density, u^i is the three-dimensional velocity of the fluid and S^{ij} is the three-dimensional stress tensor, defined so that the mechanical force per unit volume is the negative divergence of S^{ij}. If μ is to transform like P^{44} instead of being an invariant, then μ should enter into physical laws only through tensor relations involving $P^{\alpha\beta}$. This suggests that the right-hand side of equation (3.17) should be replaced by a tensor that is formed entirely from $P^{\alpha\beta}$ and $g_{\alpha\beta}$ and does not involve t. If this tensor is to remain proportional to the amount of matter present, then it should be linear in $P^{\alpha\beta}$, and equation (3.17) will be replaced by

$$\rho_{\alpha\beta} = A(P_{\alpha\beta} + BPg_{\alpha\beta}) \tag{4.2}$$

where $P_{\alpha\beta} \equiv g_{\alpha\gamma} g_{\beta\delta} P^{\gamma\delta}$, $P \equiv g_{\alpha\beta} P^{\alpha\beta}$ and A and B are undetermined constants.

Although equation (4.2) is a natural Lorentz invariant extension of Newtonian theory, it is only meaningful if all of the quantities $P^{\alpha\beta}$ are known. In Newtonian theory the masses and motions of the bodies that produce the field, as described by μ and u^i, are assumed to be known, but the internal stresses S^{ij} are not known. Instead, it is assumed in Newtonian theory that the three-dimensional geometry is Euclidean everywhere, so that ρ_{ij} must vanish in coordinates in which $x_4 = t$. The most direct Lorentz invariant extension of Newtonian theory is then described by equation (4.2) in which the quantities S^{ij} contained in $P^{\alpha\beta}$ are taken to have the particular values that will make ρ_{ij} vanish in coordinates in which $x_4 = t$. If S^{ij} is defined in this way, then the three spatial coordinates can be chosen to be Cartesian and g_{ij} will equal δ_{ij}. For a metric tensor of this form, the quantities $P_{\alpha\beta}$ are found directly from the fact that they equal $g_{\alpha\gamma}g_{\beta\delta}P^{\gamma\delta}$, where $P^{\gamma\delta}$ is given by equations (4.1a)–(4.1c), and they are

$$P_{ij} = S^{ij} + \mu(u^i + g_{i4})(u^j + g_{j4}) \tag{4.3a}$$

$$P_{i4} = S^{ij} g_{j4} + \mu(g_{4j} u^j + g_{44})(u^i + g_{i4}) \tag{4.3b}$$

$$P_{44} = g_{i4} g_{j4} S^{ij} + \mu(g_{i4} u^i + g_{44})^2 \tag{4.3c}$$

The quantity P is given by

$$P = S^{ii} + \mu(u^i + g_{i4})(u^i + g_{i4}) + \mu g \tag{4.4}$$

where g is the determinant of $g_{\alpha\beta}$ and is given by

$$g = g_{44} - g_{i4} g_{i4} \tag{4.5}$$

Putting equations (4.3a)–(4.3c) and (4.4) into the right-hand side of equation (4.2) shows that ρ_{ij} will vanish if S^{ij} is assumed to be

$$S^{ij} = -\mu(u^i + g_{i4})(u^j + g_{j4}) - B[S^{kk} + \mu(u^k + g_{k4})(u^k + g_{k4}) + \mu g]\,\delta_{ij} \tag{4.6}$$

If this is contracted and solved for S^{kk}, it is found that

$$S^{kk} = -\mu(u^k + g_{k4})(u^k + g_{k4}) - \frac{3B}{1+3B}\mu g \tag{4.7}$$

and equations (4.4) and (4.6) can be written

$$P = \frac{1}{1+3B}\mu g \tag{4.8a}$$

$$S^{ij} = -\mu(u^i + g_{i4})(u^j + g_{j4}) - \frac{B}{1+3B}\mu g\,\delta_{ij} \tag{4.8b}$$

Using these results in equations (4.3b) and (4.3c) gives

$$P_{i4} = \mu g(u^i + g_{i4}) - \frac{B}{1+3B}\mu g\, g_{i4} \tag{4.9a}$$

$$P_{44} = \mu g[2(u^i + g_{i4}) g_{i4} + g] - \frac{B}{1+3B}\mu g\, g_{i4} g_{i4} \tag{4.9b}$$

From these, the non-vanishing components on the right-hand side of equation (4.2) are found to be

$$A(P_{i4} + BPg_{i4}) = A\mu g(u^i + g_{i4}) \tag{4.10a}$$

$$A(P_{44} + BPg_{44}) = A\mu g\left[2(u^i + g_{i4}) g_{i4} + \frac{1+4B}{1+3B} g\right] \tag{4.10b}$$

In order that equation (3.17) will be the non-relativistic limit of equation (4.2), the constants A and B must be such that the right-hand sides of equations (4.10a) and (4.10b) approach the values

$$A\mu g(u^i + g_{i4}) \to 0 \tag{4.11a}$$

$$A\mu g\left[2(u^i + g_{i4})g_{i4} + \frac{1+4B}{1+3B}g\right] \to -4\pi K\mu \tag{4.11b}$$

in the limit as $c \to \infty$. In this limit the determinant g will be approximately $-c^2$, and the term $2(u^i + g_{i4})g_{i4}$ will be negligible compared to $g(1+4B)/(1+3B)$, with the result that equation (4.11b) gives

$$A = -\frac{4\pi K}{c^4}\frac{1+3B}{1+4B} \tag{4.12}$$

Since A vanishes as $1/c^4$, equation (4.11a) is automatically satisfied, and equation (4.2) becomes equation (3.17) in the non-relativistic limit, as desired. If the right-hand side of equation (4.2) is evaluated from equations (4.10a) and (4.10b) with this value of A and the left-hand side of equation (4.2) is evaluated from equation (3.12a) and (3.12b), the Lorentz invariant extension of the Newtonian field equations is found to be

$$\frac{\partial}{\partial x_j}\left(\frac{\partial g_{j4}}{\partial x_i} - \frac{\partial g_{i4}}{\partial x_j}\right) = \frac{8\pi K g}{c^4}\frac{1+3B}{1+4B}\mu(u^i + g_{i4}) \tag{4.13a}$$

$$\frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_i\,\partial x_i} - \frac{\partial^2 g_{i4}}{\partial x_4\,\partial x_i} - \frac{1}{4}\left(\frac{\partial g_{i4}}{\partial x_j} - \frac{\partial g_{j4}}{\partial x_i}\right)\left(\frac{\partial g_{i4}}{\partial x_j} - \frac{\partial g_{j4}}{\partial x_i}\right)$$
$$= -\frac{4\pi K g}{c^4}\mu\left[g + 2\frac{1+3B}{1+4B}(u^i + g_{i4})g_{i4}\right] \tag{4.13b}$$

Because it cannot be generally assumed that $\mu(u^i + g_{i4})$ vanishes, it is clear that equation (4.13a) cannot always be satisfied by metrics for which $g_{i4} = -\partial\beta/\partial x_i$, with the result that these equations lead to fields which will not be exactly of the form of equations (2.7a)–(2.7c), and hence will not be exactly Newtonian. However, the fact that equations (4.13a) and (4.13b) become the equations of Newtonian theory in the non-relativistic limit shows that the fields that they determine will usually differ only slightly from Newtonian fields. For any set of three functions g_{i4} it is always possible to find three functions β, θ and ϕ such that

$$g_{i4} = -\frac{\partial\beta}{\partial x_i} + \theta\frac{\partial\phi}{\partial x_i}$$

and for fields of physical interest it is clear that the second term on the right will be only a small correction to the first term. If a function V is then defined so that

$$g_{44} = -2V - 2\frac{\partial\beta}{\partial x_4} - c^2 + 2\theta\frac{\partial\phi}{\partial x_4}$$

then any set of metric coefficients which satisfy equations (4.13a) and (4.13b) can be written in the form

$$g_{\alpha\beta}=\sigma_{\alpha\beta}-2V\frac{\partial t}{\partial x_\alpha}\frac{\partial t}{\partial x_\beta}+\theta\left(\frac{\partial t}{\partial x_\alpha}\frac{\partial \phi}{\partial x_\beta}+\frac{\partial t}{\partial x_\beta}\frac{\partial \phi}{\partial x_\alpha}\right) \tag{4.14}$$

where

$$\sigma_{ij}=\delta_{ij}$$

$$\sigma_{i4}=-\frac{\partial \beta}{\partial x_i}$$

$$\sigma_{44}=-c^2-2\frac{\partial \beta}{\partial x_4}$$

The generalized Lorentz transformation that has been defined previously leaves the functional form of the quantities $\sigma_{\alpha\beta}$ unchanged if β is treated as an invariant function. If V, t, θ and ϕ are defined in the new Lorentz frame as invariants, it is clear that the functional form of $g_{\alpha\beta}$ given by equation (4.14) will be the same in all Lorentz frames. As a result, equation (4.2) will take the same form in all Lorentz frames when $\rho_{\alpha\beta}$ is expressed in terms of the functions β, V, t, θ and ϕ and their partial derivatives, and the entire theory will be Lorentz invariant. Because the terms in θ and ϕ do not vanish, the Euclidean nature of three-space will not be exactly preserved even under the infinitesimal transformation. However, the change in the three-dimensional geometry will be very small when the terms involving θ and ϕ are very small, as is true in most physical fields.

In addition to the fact that equations (4.13a) and (4.13b) transform in a way which is consistent with the known transformation law for a mass density and reduce to Newtonian theory in the non-relativistic limit as $c\to\infty$, they are also satisfied exactly by the exterior Schwarzschild field. To show this, it is only necessary to observe that the spherical symmetry of this field implies that g_{i4} is radial and has a magnitude that depends only on the distance from the centre of symmetry. In such a field there is always a function β such that $g_{i4}=-\partial\beta/\partial x_i$, and hence equation (4.13a) is satisfied in the exterior region where $\mu=0$. Defining V by equation (2.7c), it is seen that equation (4.13b) becomes Laplace's equation for V in the exterior region where $\mu=0$. The spherically symmetric solution of this equation is the one given in Section 2, where the parameter M must be evaluated by integrating equation (4.13b) over the region in which μ does not vanish. Carrying out this integration, and making use of equation (4.13a), leads to

$$M=\frac{1}{c^4}\int \mu g\left[g+\frac{1+3B}{1+4B}(u^i+g_{i4})g_{i4}\right]dv \tag{4.15}$$

Obviously, in the non-relativistic limit in which $g\approx -c^2$ and $c\to\infty$, $M\approx\int \mu dv$, and M is just the total mass producing the field. The exact value of M differs from this total mass only by small relativistic corrections. Since minor relativistic corrections to the value of M are much less than the probable error in estimating the mass of the sun, they will have no measurable effect on the verifiable predictions of the theory. *Thus equations* (4.13a) *and* (4.13b) *transform properly under the generalized Lorentz transformation, yield Newtonian theory in the non-relativistic limit, and also yield the Schwarzschild field, so that they lead to all of the relativistic corrections to Newtonian theory that have been verified in that field.*

5. *The Analogy to Electromagnetism*

Because most of the observed facts concerning gravity are correctly described by equations (4.13a) and (4.13b) for any value of the parameter B, there does not appear to be any way to evaluate B from gravitational observations alone. However, if it is assumed that gravity and electromagnetism are just different aspects of a unified field, it is natural to expect that they will be very similar when they are both properly formulated. In particular, since electromagnetism is linear, it seems likely that electromagnetism may be a weak-field approximation of the unified field, which suggests that the weak-field approximation of equations (4.13a) and (4.13b) should have a form very similar to the usual form of electromagnetism. It will now be shown that this analogy suggests a value for B.

The similarity between equations (4.13a) and (4.13b) and electromagnetism is already quite striking, and is made even more apparent if a three-dimensional vector $\mathbf{a}$ is defined to have the components cg_{i4} and the vectors $\mathbf{e}$ and $\mathbf{b}$ are defined by the vector relations

$$\mathbf{e} = \nabla\frac{g_{44}}{2} - \frac{1}{c}\frac{\partial \mathbf{a}}{\partial x_4} \tag{5.1a}$$

$$\mathbf{b} = \nabla x \mathbf{a} \tag{5.1b}$$

Then $\mathbf{e}$ and $\mathbf{b}$ satisfy the identical relations

$$\nabla x \mathbf{e} = -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial x_4} \tag{5.2a}$$

$$\nabla . \mathbf{b} = 0 \tag{5.2b}$$

and equations (4.13a) and (4.13b) can be written

$$\nabla x \mathbf{b} = \frac{8\pi K g}{c^3}\frac{1+3B}{1+4B}\mu\left(\mathbf{u} + \frac{1}{c}\mathbf{a}\right) \tag{5.3a}$$

$$\nabla . \mathbf{e} - \frac{b^2}{2c^2} = -\frac{4\pi K g}{c^4}\mu\left[g + \frac{2}{c}\frac{1+3B}{1+4B}\left(\mathbf{u} + \frac{1}{c}\mathbf{a}\right).\mathbf{a}\right] \tag{5.3b}$$

where $\mathbf{u}$ is the vector whose components are u^i and $b^2 \equiv \mathbf{b}.\mathbf{b}$. The motion of a particle whose position vector is $\mathbf{r}$ is given by equation (2.4), which takes the form

$$\frac{d^2\mathbf{r}}{d\tau^2} = \left(\mathbf{e} + \frac{1}{c}\frac{d\mathbf{r}}{dx_4} \times \mathbf{b}\right)\left(\frac{dx_4}{d\tau}\right)^2 - \frac{1}{c}\mathbf{a}\frac{d^2x_4}{d\tau^2} \tag{5.4}$$

If this is written with an independent variable x_4 instead of τ on the left-hand side, it becomes

$$\frac{d^2\mathbf{r}}{dx_4{}^2} = \mathbf{e} + \frac{1}{c}\frac{d\mathbf{r}}{dx_4} \times \mathbf{b} - \left(\frac{d\mathbf{r}}{dx_4} + \frac{1}{c}\mathbf{a}\right)\left(\frac{d\tau}{dx_4}\right)^2 \frac{d^2x_4}{d\tau^2} \tag{5.5}$$

The weak-field approximation of these equations is found by assuming that the field quantities can be treated as infinitesimal. In particular, it will be assumed that $(1/c)\mathbf{a}$ can be neglected relative to the particle velocity $d\mathbf{r}/dx_4$ or to the velocity $\mathbf{u}$ of the mass distribution producing the field. It will also be assumed that the term b^2 is negligible and that $g_{\alpha\beta}$ is so close to the flat-space metric that g can be approximated by $-c^2$. Also, since $\mathbf{a}$ is small, the term $(2/c)[(1+3B)/(1+4B)](\mathbf{u}+\mathbf{a}/c).\mathbf{a}$ will be neglected relative to g. Neglecting $(1/c)$ $\mathbf{a}$ relative to $d\mathbf{r}/dx_4$ in equation (5.5) and rewriting this equation with τ as independent variable gives

$$\frac{d^2\mathbf{r}}{d\tau^2} = \left(\frac{dx_4}{d\tau}\right)^2\left(\mathbf{e} + \frac{1}{c}\frac{d\mathbf{r}}{dx_4} \times \mathbf{b}\right) \tag{5.6}$$

If this is multiplied by the rest-mass density μ_0 and it is observed that the force per unit volume is defined so that it equals $\mu_0 d(d\mathbf{r}/d\tau)/dx_4$ and that the mass density μ equals $\mu_0 dx_4/d\tau$, it is seen that the force per unit volume is

$$\mu\left(\mathbf{e} + \frac{1}{c}\frac{d\mathbf{r}}{dx_4} \times \mathbf{b}\right)$$

This is exactly analogous to the Lorentz force of electromagnetism if μ is the analog of the charge density and $\mathbf{e}$ and $\mathbf{b}$ are the analogs of the electric and magnetic field intensities. The weak-field approximations of equations (5.2a), (5.2b), (5.3a) and (5.3b) are

$$\nabla x \mathbf{e} = -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial x_4} \tag{5.7a}$$

$$\nabla.\mathbf{b} = 0 \tag{5.7b}$$

$$\nabla x \mathbf{b} = -\frac{8\pi K}{c}\frac{1+3B}{1+4B}\mu\mathbf{u} \tag{5.7c}$$

$$\nabla.\mathbf{e} = -4\pi K\mu \tag{5.7d}$$

It is seen that the coefficient B, which drops out in the non-relativistic approximation, enters into the weak-field approximation, so that these equations will be close analogs of Maxwell's equations for only one value of B.

Equations (5.7a) and (5.7b) are exact analogs of two of Maxwell's equations. Equation 5.7d is closely analogous to the equation for the divergence of the electric field, and equation (5.7c) will be similarly analogous to the equation for the curl of the static magnetic field if the factor $(1+3B)/(1+4B)$ has the value $\frac{1}{2}$, that is, if $B=-\frac{1}{2}$. Using this value of B, equations (5.7a)–(5.7d) are

$$\nabla x\mathbf{e}=-\frac{1}{c}\frac{\partial \mathbf{b}}{\partial x_4} \tag{5.8a}$$

$$\nabla.\mathbf{b}=0 \tag{5.8b}$$

$$\nabla x\mathbf{b}=-\frac{4\pi K}{c}\mu\mathbf{u} \tag{5.8c}$$

$$\nabla.\mathbf{e}=-4\pi K\mu \tag{5.8d}$$

These equations are analogous to the electromagnetic equations except for the fact that the displacement current term is missing. Because of this, the theory does not predict gravitational waves, which is not too surprising for any simple extension of Newtonian theory.

The result of this analysis is a set of gravitational field equations which are Lorentz invariant and reduce to Newtonian theory in the non-relativistic limit. They also give the Schwarzschild field and all of the relativistic corrections to Newtonian theory that are found in the Schwarzschild field. In fact, it appears that the only observational evidence about the gravitational field that is not properly predicted is the existence of gravitational waves. Finally, the equations are closely analogous to the electromagnetic field equations, which is very desirable if gravity and electromagnetism are to be unified.

The resulting description of the gravitational field is given by equation (4.2), where A is given by equation (4.12) and $B=-\frac{1}{2}$, that is,

$$\rho_{\alpha\beta}=-\frac{2\pi K}{c^4}(P_{\alpha\beta}-\tfrac{1}{2}Pg_{\alpha\beta}) \tag{5.9}$$

When the stress components have been chosen so that the three-dimensional geometry is Euclidean in the three-space defined by a given value of t, these equations can be written in Cartesian coordinates with $x_4=t$ in the three-dimensional form given by equations (4.13a) and (4.13b) with $B=-\frac{1}{2}$, which is

$$\frac{\partial}{\partial x_j}\left(\frac{\partial g_{j4}}{\partial x_i}-\frac{\partial g_{i4}}{\partial x_j}\right)=\frac{4\pi Kg}{c^4}\mu(u^i+g_{i4}) \tag{5.10a}$$

$$\frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_i\,\partial x_i}-\frac{\partial^2 g_{i4}}{\partial x_4\,\partial x_i}-\frac{1}{4}\left(\frac{\partial g_{i4}}{\partial x_j}-\frac{\partial g_{j4}}{\partial x_i}\right)\left(\frac{\partial g_{i4}}{\partial x_j}-\frac{\partial g_{j4}}{\partial x_i}\right)=-\frac{4\pi Kg}{c^4}\mu[g+(u^i+g_{i4})g_{i4}] \tag{5.10b}$$

6. *The Relation to Einstein's Theory*

Einstein's field equations can be written

$$R_{\alpha\beta} = -\frac{8\pi K}{c^4}(T_{\alpha\beta} - \tfrac{1}{2}Tg_{\alpha\beta}) \tag{6.1}$$

where $T_{\alpha\beta}$ is the total stress-energy-momentum tensor, including both $P_{\alpha\beta}$ and the stress-energy-momentum tensor associated with physical fields such as the electromagnetic field. These equations differ from the extended form of Newton's theory described by equation (5.9), in that the tensor $P_{\alpha\beta}$ is replaced by $T_{\alpha\beta}$, the numerical coefficient 2 on the right-hand side is replaced by 8 and the tensor $\rho_{\alpha\beta}$ is replaced by the contracted curvature tensor $R_{\alpha\beta}$.

The first of these modifications is a very natural one from the point of view of Newtonian theory. Although the possibility that a field might produce an effective mass was not recognized at the time when Newtonian theory was developed, there is no doubt that the mass involved in Newton's theory is the total mass, regardless of whether it is 'true' mass or is the result of the energy of a field, so it is natural to assume that the stress-energy- momentum tensor in the field equations should include the stress, energy and momentum of fields as well as of ponderable matter.

The second modification, namely the difference in the numerical coefficient on the right-hand side, arises directly from the assumption that the stresses within matter are the ones that are consistent with a Euclidean three-dimensional geometry. With this assumption, equation (5.9) leads to the stresses given by equation (4.8b) with $B = -\frac{1}{2}$, which include a term $-\mu g\,\delta_{ij}$ which increases as c^2 in the limit as $c \to \infty$ because g is approximately $-c^2$. This behaviour of S^{ij} is very different from the one usually assumed in Einstein's theory, namely that S^{ij} is so small it can be neglected. The effect of this on the field equations can be seen by evaluating the quantity $P_{\alpha\beta} - \frac{1}{2}Pg_{\alpha\beta}$ approximately for large values of c. It will be sufficiently accurate for this purpose to assume that $g_{\alpha\beta}$ is equal to the flat-space metric, which vanishes off the diagonal and has diagonal values of 1, 1, 1 and $-c^2$, and to keep only terms in $P_{\alpha\beta} - \frac{1}{2}Pg_{\alpha\beta}$ which might increase at least as rapidly as c^2 when $c \to \infty$. When $P^{\alpha\beta}$ is given by equations (4.1a)–(4.1c) and $P_{\alpha\beta} \equiv g_{\alpha\gamma}g_{\beta\delta}P^{\gamma\delta}$, where $g_{\alpha\gamma}$ is the flat-space metric, and it is remembered that S^{ij} may increase as c^2 when $c \to \infty$, it is found that

$$P_{\alpha\beta} - \tfrac{1}{2}Pg_{\alpha\beta} \approx \begin{cases} S^{\alpha\beta} - \frac{1}{2}(S^{ii} - c^2\mu)\,\delta_{\alpha\beta} & (\alpha,\beta = 1,2,3) \quad (6.2a) \\ -c^2\mu u^\alpha & (\alpha = 1,2,3, \beta = 4) \quad (6.2b) \\ \dfrac{c^2}{2}(S^{ii} + \mu c^2) & (\alpha = \beta = 4) \quad (6.2c) \end{cases}$$

When the three-space is assumed to be Euclidean, so that $S^{ij} \approx c^2\mu\,\delta_{ij}$, these equations become

$$P_{\alpha\beta} - \tfrac{1}{2}Pg_{\alpha\beta} \approx \begin{cases} 0 & (\alpha,\beta = 1,2,3) \quad (6.3a) \\ -c^2\mu u^\alpha & (\alpha = 1,2,3; \beta = 4) \quad (6.3b) \\ 2c^4\mu & (\alpha = \beta = 4) \quad (6.3c) \end{cases}$$

In the situation usually considered in Einstein's theory, in which the stresses S^{ij} are assumed to be negligible, equations (6.2a)–(6.2c) become

$$P_{\alpha\beta} - \tfrac{1}{2}Pg_{\alpha\beta} \approx \begin{cases} \tfrac{1}{2}c^2\mu\delta_{\alpha\beta} & (\alpha, \beta = 1,2,3) \quad (6.4a) \\ -c^2\mu u^\alpha & (\alpha = 1,2,3; \beta = 4) \quad (6.4b) \\ \tfrac{1}{2}c^4\mu & (\alpha = \beta = 4) \quad (6.4c) \end{cases}$$

Comparing equations (6.3a)–(6.3c) with equations (6.4a)–(6.4c) it is seen that $P_{44} - \frac{1}{2}Pg_{44}$ is four times as great when the three-space is assumed to be Euclidean as it is when the stresses are assumed to be negligible. It is for this reason that the assumption that three-space is Euclidean has led to the close analogy between gravity and electromagnetism given in the previous section. However, this also requires that if $\rho_{\alpha\beta}$ is to be equated to a constant factor times $P_{\alpha\beta} - \frac{1}{2}Pg_{\alpha\beta}$, the factor must be four times as great if the stresses are neglected as it is if the three-space is assumed to be Euclidean, if both assumptions are to lead to the same non-relativistic approximation.

A further comparison of equations (6.3a)–(6.3c) and (6.4a)–(6.4c) shows that the terms $P_{i4} - \frac{1}{2}Pg_{i4}$ have the value $-c^2\mu u^i$ in both cases, at least to this approximation. However, because $P_{\alpha\beta} - \frac{1}{2}Pg_{\alpha\beta}$ is multiplied by a different coefficient in the two cases, the effect of the fluid velocity u^i in producing a gravitational field is four times as great under the usual assumptions of Einstein's theory as it is under the assumptions that have been made here. At present there does not appear to be any direct observational evidence which gives the magnitude of the effect of u^i in producing a gravitational analog of the magnetic field, but it is not impossible that such evidence may be found in the future. If so, it may give a direct indication of which set of assumptions is closer to reality.

The third way in which equation (6.1) differs from equation (5.9), namely, that $\rho_{\alpha\beta}$ is replaced by $R_{\alpha\beta}$, is suggested by the fact that the field equations then imply that $T^{\alpha\beta}{}_{;\beta} = 0$, which was interpreted by Einstein as a law of conservation of momentum and energy. It will now be shown that this substitution does not affect the non-relativistic limit of the equations, because $\rho_{\alpha\beta}$ and $R_{\alpha\beta}$ have the same limit as $c \to \infty$. To show this, it is convenient to let

$$\lambda_{\alpha\beta} \equiv R_{\alpha\beta} - \rho_{\alpha\beta} \tag{6.5}$$

Then, from equation (3.16), noting that

$$R_{\alpha\beta\gamma\delta}\, t^\beta = t_{;\alpha\delta\gamma} - t_{;\alpha\gamma\delta}$$

it is easily shown that

$$\lambda_{\alpha\delta} = -\alpha^2(t_{;}{}^{\gamma}{}_{\gamma}\, t_{;\alpha\delta} - t_{;}{}^{\gamma}{}_{\delta}\, t_{;\alpha\gamma} - t^\gamma\, t_{;\alpha\gamma\delta} + t^\gamma\, t_{;\alpha\delta\gamma}) \\ -\alpha^4\, t^\beta t^\gamma (t_{;\beta\gamma}\, t_{;\alpha\delta} - t_{;\beta\delta}\, t_{;\alpha\gamma}) \tag{6.6}$$

In coordinates in which $x_4 = t$ and $g_{ij} = \delta_{ij}$, the reciprocal of $g_{\alpha\beta}$ is given by

$$g^{ij} = \delta_{ij} + \frac{1}{g} g_{i4} g_{j4} \tag{6.7a}$$

$$g^{i4} = -\frac{1}{g} g_{i4} \tag{6.7b}$$

$$g^{44} = \frac{1}{g} \tag{6.7c}$$

and the vector $t^\alpha \equiv g^{\alpha\beta} t_{;\beta} = g^{\alpha 4}$ is given by

$$t^i = -\frac{1}{g} g_{i4} \tag{6.8a}$$

$$t^4 = \frac{1}{g} \tag{6.8b}$$

Also, in these coordinates, the second derivatives of t vanish, so that

$$t_{;\alpha\beta} = -(\alpha\beta, \gamma)\, t^\gamma \tag{6.9}$$

If the velocity of light c enters the coefficients $g_{\alpha\beta}$ only through the additive constant $-c^2$ in g_{44}, then it does not appear in the partial derivatives of $g_{\alpha\beta}$ or in $(\alpha\beta, \gamma)$. As a result, both t^α and $t_{;\alpha\beta}$ involve c only through the multiplicative factor $1/g$. Since g is approximately equal to $-c^2$, the determinant of the flat-space metric, both t^α and $t_{;\alpha\beta}$ vanish at least as rapidly as $1/c^2$ in the non-relativistic limit as $c \to \infty$. This implies that the higher covariant derivatives of t, such as $t_{;\alpha\beta\gamma}$, will also vanish as $1/c^2$. From equation (3.2) and equation (6.8b), it is seen that $\alpha^2 = -g$ in these coordinates, so that α increases like c as $c \to \infty$. From these results it is seen that the quantity in the first parenthesis on the right-hand side of equation (6.6) decreases as $1/c^4$, and when it is multiplied by α^2 the product decreases like $1/c^2$ as $c \to \infty$. Similarly, the second term on the right-hand side of equation (6.6) vanishes like $1/c^4$, so that $\lambda_{\alpha\beta}$ vanishes at least as rapidly as $1/c^2$, and $R_{\alpha\beta} \to \rho_{\alpha\beta}$ as $c \to \infty$.

The replacement of $\rho_{\alpha\beta}$ by $R_{\alpha\beta}$ is now seen to be somewhat analogous to Maxwell's addition of the displacement current term to the equations of electromagnetism. In both cases, a small relativistic correction is added to the equations to make them agree with a conservation law, and in both cases this correction leads to the prediction of waves. However, the analogy is imperfect because the conservation law used by Einstein was the conservation of momentum and energy, while the one used by Maxwell was the conservation of charge. Furthermore, there is good reason to object to the replacement of $\rho_{\alpha\beta}$ by $R_{\alpha\beta}$ because it is interpreted in Einstein's theory to imply that three-dimensional space is non-Euclidean, which is a great complication of the theory for which there appears to be no observational support. Finally, although gravitational waves appear to exist, they have not been measured with enough precision to show that they are exactly the waves predicted by Einstein's theory, and it is quite possible that there may be a much simpler way of altering the theory to include an adequate description of waves. Thus the replacement of $\rho_{\alpha\beta}$ by $R_{\alpha\beta}$ must still be regarded as a theoretical conjecture which requires more experimental confirmation. Since both Einstein's theory and Newtonian theory lead to the Schwarzschild field and Einstein's theory reduces to Newtonian theory in the non-relativistic limit, it is clear that this confirmation must come from the measurement of relativistic corrections to Newton's theory in fields other than the Schwarzschild field. Such evidence is very difficult to obtain, and until it is available, the most reliable procedure for improving gravitational theory may well be through the unification of gravity with other fields of physics, where a great deal of experimental data is already available.

7. *Conclusions*

This paper has given a description of gravity which has the advantage over Einstein's theory that it proceeds one step at a time from the most firmly established facts of Newtonian theory to the theoretical conjectures of Einstein's theory, and the degree of motivation for each step can be assessed independently of the others. The philosophical and predictive advantages of Einstein's theory have been retained by adhering to Einstein's description of gravity in terms of a curved Riemannian four-space whose geodesics are the paths of particles and light rays. However, the complexity of Einstein's formalism has been greatly reduced by retaining throughout the analysis the assumption of Newtonian theory that three-dimensional space is Euclidean, with the result that the field is described here by only four unknown metric coefficients.

Attention has first been given only to the particular class of fields given by equations (2.7a)–(2.7c), which correspond to the fields of Newtonian gravitational theory. These fields include the Schwarzschild field, so that this description of gravity not only reduces to Newtonian theory in its nonrelativistic limit but also gives all of Einstein's corrections to Newtonian theory in the Schwarzschild field. The conditions that must be met in order that a metric tensor will describe a Newtonian field in which the potential function V satisfies Poisson's equation have been formulated in generalized space-time coordinates in equation (3.17). Since equation (3.17) is an exact description of such a field, there is a strong motivation for believing that it must be very nearly satisfied by many of the most important fields that occur in nature.

The Lorentz invariance of this formulation has then been investigated by considering the transformation properties of the field equations under the generalized form of the Lorentz transformation which has been developed previously, and it is shown that equation (3.17) takes the same form in all Lorentz frames only if $K\mu$ is assumed to be an invariant function. Since it is known from the special theory that μ is not an invariant function, equation (3.17) must be modified if it is to agree with the known transformation properties of the mass density. This can be done by replacing equation (3.17) by equation (4.2) in which A and B are unspecified constants. This equation reduces to Newtonian theory in the non-relativistic limit as $c \to \infty$ if the stress components are assumed to be such that the three-dimensional geometry is Euclidean and the coefficient A is given by equation (4.12). The coefficient B cannot be determined from the non-relativistic approximation of the theory, nor can it be determined from measurements of the relativistic corrections to Newtonian theory in the exterior Schwarzschild field, because this field is an exact solution of equation (4.2) in the region in which $P_{\alpha\beta}$ vanishes. Thus there appears to be no observational evidence from which the value of B can be deduced.

However, if there is to be any real hope of unifying gravity and electromagnetism, then it is desirable to formulate gravity in a way which is not only an accurate description of the observed facts but which is also closely analogous to classical electromagnetism. The fact that electromagnetism is a linear theory suggests that it may be a weak-field approximation of the unified field, and hence should be a close analog of the weak-field approximation of gravity. Such a close analogy has been shown to exist here if $B = -\frac{1}{2}$, in which case the field equations are given by equation (5.9), which becomes equations (5.10a) and (5.10b) in coordinates in which $x_4 = t$. The motivation for these equations is very strong if it is assumed that there is a close analogy between gravity and electromagnetism, but this is an assumption of simplicity which is not based directly on observation.

One of the most important differences between this description of gravity and Einstein's theory is that the stresses have been defined here so that the three-dimensional geometry is Euclidean, whereas the stresses are usually assumed to be negligible in Einstein's theory. As a result, the quantity $P_{\alpha\beta} - \frac{1}{2}Pg_{\alpha\beta}$ on the right-hand side of equation (5.9) is approximately given by equations (6.3a)–(6.3c) instead of by equations (6.4a)–(6.4c), as would have been the case if the stresses had been neglected. This reduces the numerical coefficient of the momentum density in the field equations to approximately one-quarter of the value that it has in Einstein's theory, and thus leads to a much closer analogy between gravity and electromagnetism than exists in the weak-field approximation of Einstein's theory. An experimental measurement of the effect of a momentum density in producing a gravitational analog of the magnetostatic field has not yet been made but may become possible in the future.

The other major difference between the present description of gravity and Einstein's description is that the tensor $\rho_{\alpha\beta}$ is replaced by $R_{\alpha\beta}$ in Einstein's theory. This makes it possible for three-space to be non-Euclidean in Einstein's theory even in regions in which the total stress-energy-momentum tensor vanishes. This greatly increases the complexity of the theory in a way that is presently supported by observation only to the extent that replacing $\rho_{\alpha\beta}$ by $R_{\alpha\beta}$ is one possible way of explaining the existence of gravitational radiation. Since there may be much simpler ways of including gravitational radiation in the theory, this change must still be regarded as a theoretical conjecture.

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